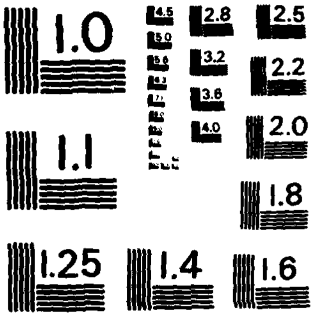
AD-A133 760 THE GAUSS-TCHEBYSHEV INEQUALITY FOR UNIMODAL DISTRIBUTIONS(U) FLORIDA STATE UNIV TALLAHASSEE DEPT OF STATISTICS S W DHARMADHIKARI ET AL. JUN 83 1/1

UNCLASSIFIED FSU-STATISTICS-M659 AFOSR-TR-83-0770 F/G 12/1 NL

END
DATE
FILMED
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

AD-A133760

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: AFOSR-TR- 83-0770
2. GOVT ACCESSION NO.: AD-A133760
3. RECIPIENT'S CATALOG NUMBER:
4. TITLE (and Subtitle): Document describes "THE-GAUSS-TCHEBYSHEV INEQUALITY FOR UNIMODAL DISTRIBUTIONS", # 1473 ...
5. TYPE OF REPORT & PERIOD COVERED: Annual
6. PERFORMING ORG. REPORT NUMBER:
7. AUTHOR(s): S.W. Dharmadhikari, Kumar Joag-dev
8. CONTRACT OR GRANT NUMBER(s): F49620-82-K-0007
9. PERFORMING ORGANIZATION NAME AND ADDRESS: The Florida State University, Department of Statistics, Tallahassee, Florida 32306
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: PE61102F; 2304/A5
11. CONTROLLING OFFICE NAME AND ADDRESS: Math & Info. Sciences, AFOSR/NM, Bolling AFB DC 20332
12. REPORT DATE: June 1983
13. NUMBER OF PAGES: 13
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office):
15. SECURITY CLASS. (of this report): UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE:

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION ST... ...ENT (of ... abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY ...TES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Gauss' inequality, Tchebyshev's inequality, Markov's inequality, unimodal distribution, convex structure.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

DD FORM 1 JAN 73 1473

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

Let X be a random variable whose distribution is unimodal with mean μ. For $r > 0$, let $\lambda_r = \{E|X - \mu|^r\}^{1/r}$. In this paper, we determine a value k_r such that

$$P(|X - \mu| \geq k\lambda_r) \leq [r/(r + 1)]^r . k^{-r},$$

The determination of a value

for all $k \geq k_r$. This improves and extends a recent result of Vysochanskii and Petunin (1979) who have only considered the case $r = 2$ with a higher value for k_2. ~~Our~~ The proof is also considerably simpler because it uses the convex structure of the class of unimodal distributions.

AFOSR-TR- 83-0770

The Gauss-Tchebyshev Inequality for Unimodal Distributions

by

S. W. Dharmadhikari and Kumar Joag-dev[1]

Southern Illinois University and University of Illinois and Florida State University

FSU Statistics Report No. M-659

AFOSR Technical Report No. 83-157

June, 1983

The Florida State University
Department of Statistics
Tallahassee, Florida 32306

AMS 1980 Subject Classification: 60 E 15

Key Words and Phrases. Gauss' inequality, Tchebyshev's inequality, Markov's inequality, unimodal distributions, convex structure.

[1]Research sponsored by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under Grant Number F49620 [redacted] 82-K-0007. The U.S. Government is authorized to reproduce and distribute reprints for Governmental purposes notwithstanding any copyright notation thereon.

83 10 04 250

The Gauss-Tchebyshev Inequality for Unimodal Distributions

by

S. W. Dharmadhikari and Kumar Joag-dev

Summary

Let X be a random variable whose distribution is unimodal with mean μ. For $r > 0$, let $\lambda_r = \{E|X - \mu|^r\}^{1/r}$. In this paper, we determine a value k_r such that

$$P(|X - \mu| \geq k\lambda_r) \leq [r/(r + 1)]^r . k^{-r},$$

for all $k \geq k_r$. This improves and extends a recent result of Vysochanskii and Petunin (1979) who have only considered the case $r = 2$ with a higher value for k_2. Our proof is also considerably simpler because it uses the convex structure of the class of unimodal distributions.

AMS 1980 Subject Classification: 60 E 15

Key Words and Phrases. Gauss' inequality, Tchebyshev's inequality, Markov's inequality, unimodal distributions, convex structure.

1. Introduction.

Let X be a real random variable with mean μ and let $r > 0$. Markov's inequality states that, for every given a and every $k > 0$,

$$P(|X - a| \geq k) \leq E(|X - a|^r)/k^r. \tag{1.1}$$

If $a = \mu$ and $r = 2$, (1.1) reduces to the usual Tchebyshev inequality. Suppose now that the distribution of X is unimodal with a mode M. A result attributed to Gauss (1821) states that

$$P(|X - M| \geq k) \leq (4/9)\ E\ (|X - M|^2)/k^2, \tag{1.2}$$

for all $k > 0$. In other words, if $a = M$, the bound on the right side of (1.1) can be reduced by a factor (4/9) when $r = 2$. As a consequence, if the distribution of X is both symmetric and unimodal, then $M = \mu$ and (1.2) gives

$$P(|X - \mu| \geq k) \leq 4\sigma^2/(9k^2), \tag{1.3}$$

where $\sigma^2 = \mathrm{Var}\ (X)$. Recently, Vysochanskii and Petunin (1979) showed that (1.3) is valid without the assumption of symmetry as long as $k \geq \sqrt{8/3}$. In this paper, we first obtain the factor by which the bound in (1.1) can be improved if the distribution is unimodal and $a = M$. We then show that the improved bound is valid even if $a = \mu$ as long as k is suitably large. For $r = 2$, we need $k \geq \sqrt{19/3}$, which is better than the value $\sqrt{8/3}$ obtained by Vysocanskii and Petunin.

2. Preliminaries.

In this section we give some results on certain convex sets of distributions.

DEFINITION 2.1. A distribution function F is said to be <u>unimodal</u> about a mode M if F is convex on $(-\infty, M)$ and concave on (M, ∞).

Let C_M denote the set of all distributions on R that are unimodal about M. Then C_M is clearly convex (under mixtures). It is also closed under weak convergence; see Gnedenko and Kolmogorev (1968), Section 32. Let U_M denote the set of all uniform distributions on intervals with M as one end point. Then C_M is the closed convex hull of U_M. Another equivalent statement of this result is as follows; [see Feller (1971), p. 158].

THEOREM 2.1. *A random variable* X *has a unimodal distribution with mode* M *if, and only if,* X *is distributed as* M + UZ, *where* U *is uniform on* (0, 1) *and* U, Z *are independent.*

This theorem enables one to reduce many problems involving unimodal distributions to those involving uniform distributions.

Let D_μ denote the set of all distributions on R which have mean μ and finite support. The following lemma is possibly known.

LEMMA. 2.1. *Every distribution in* D_μ *is a finite convex mixture of one or two point distributions with mean* μ.

Proof. Without loss of generality, let $\mu = 0$. Let $P \in D_0$ and let ν be the size of the support of P. The lemma holds if $\nu \le 2$. Suppose the lemma holds for $\nu \le n$, where $n \ge 2$. Let Y be a random variable with distribution P and suppose Y takes exactly $(n + 1)$ values. Since Y is not degenerate and $E(Y) = 0$, we can find $a > 0$ such that

$$\xi = P(Y = -a) > 0 \text{ and } \eta = P(Y = b) > 0.$$

Without loss of generality, assume that $a\xi \ge b\eta$. Consider the two-point distribution P_0 which puts mass $a/(a + b)$ at the point b mass $b/(a + b)$ at the point $(-a)$. Then P_0 has zero mean and

$$P = \alpha P_0 + (1 - \alpha)P_1, \tag{2.1}$$

where $\alpha = \eta(a + b)/a$. Note that αP_0 accounts for all the mass at b. It is clear that $\alpha > 0$. On the other hand, since Y takes at least 3 values, we must have $\xi + \eta < 1$. Therefore

$$\eta(a + b) = a\eta + b\eta \le a\eta + a\xi = a(\xi + \eta) < a. \quad \text{Thus } \alpha < 1.$$

The quantity P_1 in (2.1) is a distribution which puts positive mass at $\le n$ points, since the mass at b is accounted for by αP_0. By the induction hypothesis, P_1 is expressible as a mixture of one or two point distributions with zero mean. Therefore, by (2.1), P also can be expressed as a mixture of the required type. The proof of the lemma is now complete.

The following lemma is standard.

LEMMA 2.2. *Let* $r > 0$ *and let* X *be a real random variable with* $E(|X|^r) < \infty$. *Then we find a sequence of random variables* X_n *such that each* X_n *takes only a finite number of values and* $E(|X_n - X|^r) \to 0$. *Moreover, if* $r \ge 1$, *then we can shoose the* X_n *in such a way that* $E(X_n) = E(X)$ *for all* n.

3. The Gauss-Tchebyshev inequality.

The Markov inequality states that

$$P(|X - a| \ge k) \le E(|X - a|^r)/k^r, \tag{3.1}$$

where X is a real random variable, $a \in R$, $r > 0$ and $k > 0$. If $a = E(X)$ and $r = 2$, (3.1) gives the usual Tchebyshev inequality. If X has a distribution which is unimodal about M, then the bound on the right side of (3.1) can be reduced by a factor which depends on r. This is made precise by Theorem 3.1. below. For the special case $r = 2$, Theorem 3.1 goes back to Gauss (1821).

THEOREM 3.1. *Let* X *have a distribution which is unimodal about* M. *Then for every* $r > 0$ *and every* $k > 0$,

$$P(|X - M| \geq k) \leq \left(\frac{r}{r+1}\right)^r \frac{(E(|X - M|^r)}{k^r}. \tag{3.2}$$

Moreover, this bound is sharp.

Proof. Without loss of generality, let $M = 0$. Since (3.2) is trivially true if $E|X|^r = \infty$, we assume that $E|X|^r < \infty$. Since X is unimodal about zero, by Theorem 2.1, X has the same distribution as UZ, where U is uniform on (0, 1) and U, Z are independent. Now $E|X|^r = E(|Z|^r)/(r + 1)$. Therefore $E|Z|^r < \infty$. Lemma 2.2 shows that it is sufficient to establish (3.2) in the case where Z takes only a finite number of values. Now the set of distributions of Z, for which (3.2) is valid, is clearly convex. Therefore we need only consider the case where Z is degenerate. Finally, (3.2) is clearly unaffected by a change of scale. Therefore we may and do assume that Z is degenerate at 1, so that X has the uniform distribution on (0, 1). In this case, $E|X|^r = 1/(r + 1)$ and

$$P(|X| \geq k) = \begin{cases} (1 - k), & \text{if } 0 < k \leq 1 \\ 0 \quad , & \text{if } k \geq 1. \end{cases}$$

Therefore

$$k^r P(|X| \geq k) = \begin{cases} k^r(1 - k) & \text{if } 0 < k \leq 1 \\ 0 \quad , & \text{if } k \geq 1. \end{cases}$$

For fixed r, the last quantity becomes maximum when $k = r/(r + 1)$. The maximum value is $r^r/(r + 1)^{r+1}$. Therefore

$$k^r P(|X| \geq k \leq \left(\frac{r}{r+1}\right)^r \cdot \frac{1}{(r+1)} = \left(\frac{r}{r+1}\right)^r E|X|^r,$$

which proves (3.2). Further the above calculation shows that the bound is sharp.

The special case $r = 2$ gives the Gauss inequality.

COROLLARY 3.1. (Gauss). *If* X *has a distribution which is unimodal about* M, *then, for all* $k > 0$,

$$P(|X - M| \geq k) \leq 4\, E(|X - M|^2)/(9k^2)$$

COROLLARY 3.2. *Let* X *have a symmetric and unimodal distribution. Let* $\mu = E(X)$ *and* $\sigma^2 = \mathrm{Var}(X)$. *Then, for all* $k > 0$,

$$P(|X - \mu| \geq k\sigma) \leq 4/(9k^2). \tag{3.3}$$

Proof. Immediate from corollary 3.1, because $M = \mu$.

Recently, Vysochanskii and Petunin (1979) showed that (3.3) holds for unimodal random variables without the assumption of symmetry provided that $k \geq \sqrt{(8/3)}$. We improve and generalize their results below (Theorem 3.2). Our proof is also considerably simpler because we use the convex structures introduced in Section 2.

THEOREM 3.2. *Let* X *have a unimodal distribution with mean* μ. *Let* $\tau_r = E(|X - \mu|^r)$. *Then, for every* $k > 0$,

$$P(|X - \mu| \geq k) \leq \max\left[\frac{(r+1)\tau_r - k^r}{rk^r}, \left(\frac{r}{r+1}\right)^r \frac{\tau_r}{k^r}\right].$$

Proof. Without loss of generality assume that $\mu = 0$. Suppose X is unimodal about M. If 0 is also a mode of X, then the theorem follows from Theorem 3.1. So, suppose that X is not unimodal about 0. Again, we may assume that $M > 0$. By Theorem 2.1, X has the same distribution as $M + UZ$, where U is uniform on $(0, 1)$ and U, Z are independent. Now $0 = E(X) = M + \frac{1}{2}E(Z)$.

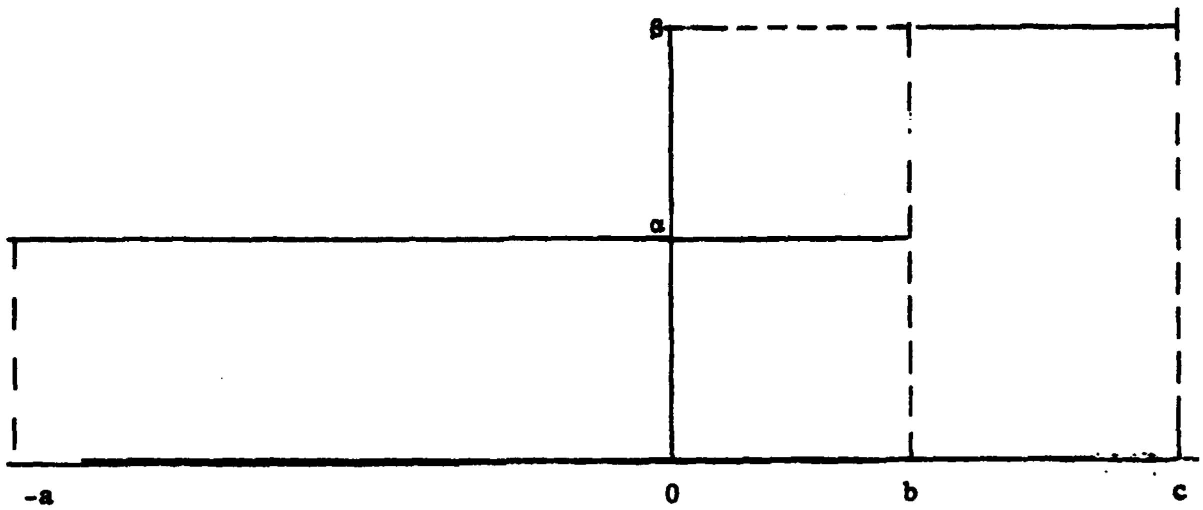


Fig. 1. Graph of the density f in the proof of Theorem 3.2.

Therefore $E(Z) = -2M$. It is clear from Lemma 2.2 that it is sufficient to prove the theorem in the case where Z takes only a finite number of values. Moreover, since the mean of X is fixed at 0, the class of distributions of X for which the theorem holds is convex. Therefore the second assertion of Lemma 2.2 and Lemma 2.1 show that it is sufficient to prove the theorem in the case where Z takes exactly two values. We have thus reduced our problem to the case where X has the density f given by

$$f(x) = \begin{cases} \alpha \quad , \text{ if } -a < x < b, \\ \beta \quad , \text{ if } \quad b < x < c, \\ 0 \quad , \text{ elsewhere} \end{cases}$$

Here a, b, c are suitable positive constants. A graph of f is given in Fig. 1. Since f is not to be unimodal about 0, we must have $\alpha < \beta$. Further the condition $E(X) = 0$ requires that $b < c < a$. Three cases arise.

Case 1. Suppose $0 < k < b$. Here $P[|X| < k] = 2\alpha k$ and so

$$\int_{|t|<k} |t|^r f(t)dt = \frac{2\alpha k^{r+1}}{(r+1)} = \frac{k^r P[|X| < k]}{(r+1)} . \tag{3.4}$$

Case 2. Suppose $b < k < c$. Here

$$P[|X| < k] = \alpha(b + k) + \beta(k - b),$$

and

$$\int_{|t|<k} |t|^r f(t)dt = \frac{\alpha(b^{r+1}+k^{r+1}) + \beta(k^{r+1}-b^{r+1})}{(r + 1)} .$$

Simple algebraic manipulations yield

$$(3.5)\qquad \int_{|t|<k} |t|^r f(t)dt - \frac{k^r P[|X|<k]}{(r+1)} = \frac{b(\beta-\alpha)\ (k^r-b^r)}{(r+1)}\ .$$

Since $\alpha < \beta$ and $0 < b < k$, the right side of (3.5) is positive.

Consider the two cases together. That is, let $0 < k < c$. Then (3.4) and (3.5) show that

$$(3.6)\qquad \int_{|t|<k} |t|^r f(t)dt \geq \frac{k^r P[|X|<k]}{(r+1)}\ .$$

Now

$$\tau_r = E|X|^r = \int_{|t|\geq k} |t|^r f(t)dt + \int_{|t|<k} |t|^r f(t)dt$$

$$\geq k^r P[|X|\geq k] + \frac{k^r P[|X|<k]}{(r+1)}\ , \quad \text{[using (3.6)]}.$$

Writing $P[|X| < k] = 1 - P[|X| \geq k]$, we get

$$\tau_r \geq k^r\ [(\frac{r}{r+1})\ P[|X| \geq k] + \frac{1}{(r+1)}],$$

Therefore

$$(3.7)\qquad P[|X| \geq k] \leq \frac{(r+1)\tau_r - k^r}{rk^r}$$

Case 3. Suppose that $c < k$. Define a new density g as follows.

$$g(x) = \begin{cases} \gamma\ , & \text{if } 0 < x < c, \\ f(x)\ , & \text{elsewhere} \end{cases}$$

Since g agrees with f outside the interval (0, c), the constant γ must satisfy

$$(3.8) \qquad \gamma c = \int_0^c f(t)dt = \alpha b + \beta(c-b).$$

Now let $\delta_r = \int_{-\infty}^{\infty} |t|^r g(t)dt$. Then

$$(r+1)\ (\tau_r - \delta_r) = (r+1)\ [\int_0^c t^r f(t)dt - \int_0^c t^r g(t)dt]$$

$$= \alpha b^{r+1} + \beta(c^{r+1} - b^{r+1}) - \gamma c^{r+1}$$

$$= \alpha b^{r+1} + \beta(c^{r+1} - b^{r+1}) - c^r[\alpha b + \beta(c-b)], \text{ [using (3.8)]}$$

$$= b(\beta-\alpha)\ (c^r - b^r).$$

Since $\alpha < \beta$ and $0 < b < c$, we see that $\delta_r \le \tau_r$. Let Y be a random variable with density g. Since g is unimodal about 0, Theorem 3.1 shows that

$$P(|Y| \ge k) \le (\frac{r}{r+1})^r \frac{\delta_r}{k^r} \le (\frac{r}{r+1})^r \frac{\tau_r}{k^r} \ .$$

But since $k > c$, the densities g and f agree on the set $(-\infty, -k] \cup [k, \infty)$. Therefore

$$(3.9) \qquad P[|X| \ge k] = P[|Y| \ge k] \le (\frac{r}{r+1})^r \frac{\tau_r}{k^r} \ .$$

The theorem now follows from (3.7) and (3.9).

COROLLARY 3.3. <u>Let</u> X <u>be a unimodal random variable with mean</u> μ. <u>Let</u> $\lambda_r = \{E(|X-\mu|^r)\}^{1/r}$. <u>Then</u>, <u>for every</u> $k > 0$,

$$P[|X-\mu| \ge k\lambda_r] \le \max \left\{ \frac{(r+1)-k^r}{rk^r}, \quad [\frac{r}{(r+1)k}]^r \right\} .$$

Proof. Immediate from Theorem 3.2, if we replace k by $k\lambda_r$ and note $\lambda_r^r = \tau_r$. Observe that

$$\frac{(r+1)-k^r}{r} \le \left(\frac{r}{r+1}\right)^r \text{ whenever } k \ge k_r, \text{ where}$$

$$k_r = \left[\frac{(r+1)^{r+1} - r^{r+1}}{(r+1)^r}\right]^{1/r} . \tag{3.10}$$

Therefore, the following corollary is immediate.

COROLLARY 3.4. *With the same notation as in Corollary* 3.3,

$$P(|X - \mu| \ge k\lambda_r) \le \left(\frac{r}{r+1}\right)^r k^{-r} ,$$

for all $k \ge k_r$, *where* k_r *is given by* (3.10).

For a comparison of our results with those given by Vysochanskii and Petunin, we write the special cases of the last two corollaries when $r = 2$.

COROLLARY 3.5. *Let* X *be a unimodal random variable with mean* μ *and variance* σ^2. *Then, for every* $k > 0$,

$$P(|X - \mu| \ge k\sigma) \le \max\left[\frac{3-k^2}{2k^2}, \frac{4}{9k^2}\right] . \tag{3.11}$$

Consequently, for every $k \ge \sqrt{19}/3$,

$$P(|X - \mu| \ge k\sigma) \le \frac{4}{9k^2} . \tag{3.12}$$

Proof. We only need to note that $k_2 = \sqrt{19}/3$.

REMARK. The inequality (3.11) is an improvement of the result of Vysochanskii and Petunin (1979). They have $(4-k^2)/3$ in place of our $3-k^2/2$. Consequently, they prove (3.12) for all $k \ge \sqrt{8/3}$.

It is to be noted that (3.12) does not hold for all $k > 0$, if the distribution is not symmetric. The following detailed analysis of the example considered by Vysochanskii and Petunin shows that (3.12) can fail if $k = 1.385$. We note that $1.385 < \sqrt{19/3}$.

EXAMPLE 3.1. Let $a \geq 1$ and consider a random variable X such that $P(X = 1) = (a - 1)/(a + 1)$ and

$$P(X \leq x) = 2(x + 1)/(a + 1)^2 \text{ , } -a < x < 1.$$

It is easy to check that $\mu = E(X) = 0$ and $\sigma^2 = \mathrm{Var}(X) = (2a - 1)/3$. Now

$$P(|X| \geq 1) = \frac{a - 1}{a + 1} + \frac{2(a - 1)}{(a + 1)^2} = \frac{(a - 1)\ (a + 3)}{(a + 1)^2} .$$

We now set $k\sigma = 1$. That is, $k = (1/\sigma)$. Then

$$k^2 P(|X \quad \mu| \geq k\sigma) = \sigma^{-2}\ P(|X| \geq 1)$$

$$= \frac{3(a - 1)\ (a + 3)}{(2a - 1)\ (a + 1)^2} = g(a), \text{ say}$$

The condition $g(a) > (4/9)$ reduces to.

$$(3.13) \qquad 8a^3 - 15a^2 - 54a + 77 < 0.$$

Numerical calculations show that (3.13) holds for $1.2816 \leq a \leq 3.05$. Since $k = \sigma^{-1}$, we see that (3.12) can fail if $.767 \leq k \leq 1.385$.

REFERENCES

[1] Feller, W. (1971). An Introduction to Probability Theory and Its Applications, Vol. II (2nd Ed.). Wiley, New York.

[2] Gauss, C. F. (1821).

[3] Gnedenko, B. V. and Kolmogorov, A. N. (1968). Limit Distributions for Sums of Independent Random Variables. Revised Edition. Addison-Wesley.

[4] Vysochanskii, D. F. and Petunin, Ju. I. (1979). Justification of the 3σ rule for unimodal distributions. Theor. Probability and Math. Statist. 21, 25-36.